

COVERS COUNTING VIA FEYNMAN CALCULUS

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ABSTRACT. Let G be a finite group. In this paper we present a tool for counting the number of principle G -bundles over a surface. As an application, we express (non-standard) generating functions for double Hurwitz numbers as integrals over commutative Frobenius algebras, associated with symmetric groups.

1. INTRODUCTION

This note is devoted to a partial case of the following big problem.

Big problem: Let $M = \{\mu_1, \dots, \mu_k\}$ be a finite ordered collection of conjugacy classes of a finite group G . Count the weighted number of principle G -bundles over a closed oriented surface Σ of genus g with k marked points q_1, \dots, q_k , such that the holonomy around q_i is in the class μ_i for $i = 1, \dots, k$.

Here the “weighted” means that we count each bundle with a weight reciprocal to the number of its automorphisms.

It is the problem of the study of a certain 2-dimensional topological quantum field theory (see, for example [1]). The corresponding numbers are correlation functions in this theory.

An efficient way to deal with it is to use its connection with the combinatorics and representation theory of the group G . Namely, if $\#N(M)$ is the number of homomorphisms of the fundamental group of $\Sigma/\{q_1, \dots, q_k\}$ to G , such that the image of an element corresponding to the complete revolution around q_i is contained in the conjugacy class corresponding to μ_i , then the numbers of bundles is the ratio of $\#N(M)$ and the order of G . For the details see [10].

We use this connection for counting the number of bundles of a special kind. Namely, denote by $h_\tau(\mu, \nu; r)$ the weighted numbers of principle G -bundles over a sphere with a non-trivial holonomy around $r + 2$ points q_0, \dots, q_{r+1} , such that the holonomy around point q_0 is in the conjugacy class μ , the holonomy around q_{r+1} is in the conjugacy class ν , and the holonomy around all the other points is in the fixed conjugacy class τ .

The main result of this note is the following. We construct generating functions for the numbers $h_\tau(\mu, \nu; r)$ as integrals over the center of the group algebra $\mathbb{C}[G]$. Namely, for an explicitly constructable square matrix $A_\tau(\beta)$, we have

$$\sum_{k=0}^{\infty} h_\tau(\mu, \nu; r) \beta^r = \frac{1}{Z_\tau} \frac{\text{tr}(f_{\mu^{-1}} f_\mu) \text{tr}(f_{\nu^{-1}} f_\nu)}{\#G} \int_{Z\mathbb{C}[G]} z_\mu \bar{z}_\nu e^{-\langle A_\tau(\beta) z, z \rangle} dm,$$

where all the notation is explained in the text. This presentation allows us to express the generating functions for numbers $h_\tau(\mu, \nu; r)$ as entries of the matrix $(A_\tau(\beta))^{-1}$, multiplied by certain constants.

As an corollary of this presentation, we find that the corresponding generating functions are rational functions in β . We also derive a non-linear differential equation for this functions.

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The main example for our considerations are (disconnected) double Hurwitz numbers. It is the case when the group G is a symmetric group S_d , and the distinguished class τ is the class of a transposition. Principle S_d -bundles are just ramified covers of degree d . These numbers have a very rich structure related with various fields of mathematics.

In the case when one of the holonomies (say, ν) is in the conjugacy class of the unit, the celebrated ELSV-formula (see, for example the original paper [5]) relates the numbers of connected covers with intersection numbers on moduli space of complex curves.

The geometrical meaning of double Hurwitz numbers is not yet completely understood. In the paper [8], the authors conjectured that if $\nu = (d)$, there exists a moduli space $Pic_{g,n}$ such that this numbers can be expressed in the terms of intersection numbers on some properly chosen compactification. They have also found an expression for generating functions for double Hurwitz numbers in the terms of Schur polynomials and found an explicit formula for one-part double Hurwitz numbers (it is the case when there is a complete branching over one of the special points). In the paper [12] there was shown that a generating function for double Hurwitz numbers is a τ -function for Toda lattice hierarchy.

Later in [3], it was found that the computation of double Hurwitz numbers can be carried out in the language of tropical geometry. In [2] the authors give a method for computation of number of covers with an arbitrary ramification. In a sense, our work is a continuation of the research carried out in [2].

The paper [7] provides the following formula for a generating function for the double Hurwitz numbers, generalizing the results of [8]. Let the cyclic type of μ be $(\mu_i)_{i=1}^{l(\mu)}$, the cyclic type of ν be $(\nu_j)_{j=1}^{l(\nu)}$. By $\varsigma(z)$ we denote the following function

$$\varsigma(z) = e^{z/2} - e^{-z/2}.$$

Then, for μ and ν satisfying certain condition (the pair (μ, ν) is contained in a *chamber* of $R_{l(\mu), l(\nu)}$, for the details see [7]) we have

$$\sum_{r=0}^{\infty} h_{\tau}(\mu, \nu; r) \frac{z^r}{r!} = \frac{1}{(\#Aut \mu) \prod \mu_i} \frac{1}{(\#Aut \nu) \prod \nu_j} \frac{1}{\varsigma(dz)} \sum_{k=1}^{t(\mu, \nu)} \prod_{m=1}^{l(\mu)+l(\nu)+1} \varsigma(z Q_{k,l}^{\mu, \nu}),$$

where $t(\mu, \nu)$ is a finite number, and $Q_{k,l}^{\mu, \nu}$ are certain quadratic polynomials in μ_i and ν_j .

This result is obtained using the the infinite wedge space formalism.

In the language of the infinite wedge space, the application of our technique to the problem of double Hurwitz numbers computation is rather trivial. Namely, we study the vacuum expectations of operators of the form

$$\langle \prod \alpha_{\mu_i} (1 - \beta \mathcal{F}_2)^{-1} \prod \alpha_{-\nu_j} \rangle,$$

which is nothing more but a study of entries of the block-diagonal operator $(1 - \beta \mathcal{F}_2)^{-1}$ in the basis $\prod \alpha_{-\nu_j} |0\rangle$.

Our technique allows to produce generating functions for double Hurwitz numbers even in the case when a pair (μ, ν) belongs to a *resonance arrangement* in the terminology of [7]. Also it works in the case of an arbitrary finite group.

The paper is organized as follows. A presentation of a fundamental group of a surface is constructed in the section 2. Section 3 presents a reformulation of the enumerational problem in the algebraic language. Section 4 is a short overview of Feynman calculus, and section 5 is devoted to the application of this technique to our problem. Section 6 contains the calculation of (disconnected) double Hurwitz numbers for the degrees $d = 2, 3, 4$. We also give a comparison of our generating function with the generating function of [7] in one particular case.

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2. GROUP PRESENTATION CORRESPONDING TO A PAIR-OF-PANTS DECOMPOSITION OF A SURFACE

We begin with a special presentation of the fundamental group of a punctured surface.

Definition 2.1. *Let g and n be two non-negative integer numbers such that $2 - 2g - n < 0$. Denote by $H_{g,n}$ the fundamental group of a connected oriented surface of genus g with n punctures.*

Let us define some notions we are going to use for construction of a presentation of $H_{g,n}$.

Definition 2.2. *By a 1-3-valent graph Γ we mean a finite graph having only vertices of valence 1 and 3. The first Betti number of Γ (considered as a one dimensional cell complex) is referred to as the genus. The 1-valent vertices of a graph are referred to as ends; the edges, incident to ends, are called leaves; all the other edges are called inner edges. The set of all vertices of Γ which are not ends is called the set of inner vertices and is denoted by $V^0(\Gamma)$. The set of all edges of Γ is denoted by $E(\Gamma)$; the set of inner edges of Γ is denoted by $E^0(\Gamma)$. The set of the edges of Γ adjacent to $v \in V^0(\Gamma)$ is denoted by E_v .*

From now on, all the graphs we are dealing with, are supposed to be 1-3-valent, if the contrary is not stated explicitly.

Another notion we are going to use is the following.

Definition 2.3. *Let Γ be a connected graph. A maximal tree T in Γ is a connected subgraph of Γ of genus 0, such that the set of vertices of T coincides with the set of vertices of Γ .*

Each connected graph contains a maximal tree.

Definition 2.4. *An enhanced graph is a connected graph enhanced with the following additional data:*

- *A choice of an orientation on all the edges of Γ ;*
- *A choice of a cyclic order on a set of half-edges, adjacent to every 3-valent vertex of Γ ;*
- *A choice of a maximal tree T in Γ ;*
- *A choice of a basepoint p in T .*

For a given edge $e \in E(\Gamma)$, and a given orientation on e , a vertex v adjacent to e is referred to as a source of e , if the orientation of e is directed outwards with respect to v . If the orientation of e is directed inwards with respect to v , the vertex v is referred to as a sink of e .

We use the same notation for an enhanced graph and its underlying graph if it does not lead to an ambiguity.

The starting point of the following construction is an enhanced graph Γ with n ends and the genus equal to g . Our construction of a presentation of $H_{g,n}$ consists of the following steps.

- (1) Consider Γ as a one-dimensional cell complex. Subdivide each inner edge of Γ by a 2-valent vertex. Glue an additional 1-cell, in such a way that both boundary components get glued to the same point, to each of the newly

added 2-valent vertices, and to each of the ends of Γ . All the newly added 1-cells are referred to as *circles*. Choose an orientation on all the circles in an arbitrary way. The obtained cell complex is homotopy equivalent to a bouquet of $|E(\Gamma)| + g$ circles. Its fundamental group is a free group of the corresponding rank. We present it in the following way.

- *Generators corresponding to edges in T* are presented by based loops that start in p , go along edges of T to the point of the attachment of the circle corresponding to the edge we are interested in, make a complete revolution along it in the positive direction, and return back to p along edges of M .
- *Generators corresponding to edges that are not in T* are presented by based loops with the base p . According to the chosen orientation, each edge that does not belong to T has a beginning and an end. Start in p , go along edges of T to the beginning of the edge we are interested in, go along this edge to the point of attachment of the circle, make a full revolution in the positive direction, and return back the same way.

If e is an edge of Γ , the described above generator corresponding to e is denoted p_e .

- *Generators corresponding to nontrivial cycles in Γ* are presented by based loops with base p . The set of such generators is in the bijection with the set of edges of Γ that does not belong to T . Start in p , go along edges of M to the beginning of the edge which is not in T , continue along it to the end, and return back to p along edges of T .

If e is an edge of Γ which is not contained in the maximal tree, the above described generator is denoted by g_e .

- (2) For each of the 3-valent vertices of Γ prepare a 2-cell with an oriented boundary. Attach them to the cell complex in such a manner that the boundary goes in the following way. We begin in a 3-valent vertex, go along one of the edges attached to it to the point of the attachment of the circle, and go along it using the following convention: if we came there along the orientation of the edge, we go along the circle in the positive direction. Otherwise, we go along it in the negative direction. Then we return back to the 3-valent vertex we have started with and continue along a next half-edge (according to the chosen cyclic order on the set of half-edges) adjacent to it. Repeating the procedure for every adjacent edge, we obtain the way each of the 2-cells is glued to the cell complex. For an example of such a gluing, see the figure 1.

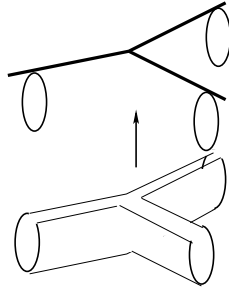


FIGURE 1. An example of gluing a 2-cell to a graph with attached circles. A vicinity of a 3-valent vertex of a graph is shown in the top part of the picture.

The obtained cell complex is homotopy equivalent to a connected oriented surface of genus g with n punctures. Each of the attached 2-cell provides a relation in its fundamental group. Note, that for every edge e not containing in the maximal tree, the generator g_e appears in only one of the obtained relations. Van Kampen theorem implies that these relations are defining relations in $H_{g,n}$.

Euler characteristics of a genus g connected graph is $1 - g$. It allows us to compute explicitly the number of generators and relations in the presentation in terms of the genus and the number of branching points. Therefore, we have just proved the following Lemma.

Lemma 2.5. *The group $H_{g,n}$ admits a presentation with $4g - 3 + 2n$ generators and $2g - 2 + n$ relations.*

Each 2-cell used in the construction corresponds to a pair of pants. So the obtained presentation corresponds to a pair-of-pants decomposition of a surface.

Example: Let us consider the enhanced graph on the figure 2 with the blackboard cyclic ordering of half-edges in its vertices. The maximal tree is chosen as it is depicted. The basepoint is the only 3-valent vertex in the maximal tree.

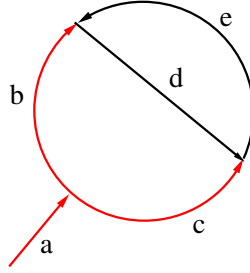


FIGURE 2. An enhanced graph. The maximal tree is shown in red. The orientation of the edge is shown by an arrowhead. The cyclic ordering of half-edges obey the blackboard convention. The basepoint is the only 3-valent vertex of the maximal tree.

The corresponding group presentation is the following:

$$H_{2,1} = \langle p_a, p_b, p_c, p_d, p_e, g_d, g_e \mid p_a^{-1} p_b p_c = p_b^{-1} g_e^{-1} p_e g_e p_d = p_c^{-1} g_d^{-1} p_d g_d p_e = 1 \rangle.$$

3. PROBLEM REFORMULATION IN THE TERMS OF GROUP ALGEBRA

The problem of principle G -bundles enumeration is equivalent to the problem of enumeration of morphisms from the fundamental group of a punctured surface to a finite group G (for the details see, for example [10]).

The number of possible morphism between two groups does not depend on their presentations. So, considering the presentation constructed in the previous section, we split the main problem of this note in two parts:

- Fix the conjugacy classes of *all* the images of the generators of $H_{g,n}$ corresponding to the edges of the chosen graph Γ , and count the number of such morphisms.
- Take a sum over all possible choices of conjugacy classes of images of the inner edges.

A similar method was discussed in [2], where the authors consider the problem from the tropical point of view. In a sense, from one hand, the main result of this paper is the generalization of the result of [2] on the case of an arbitrary finite

group. From the other hand, it is nothing else but a well-known argument, which can be found, for example, in [4], but in a slightly non-standard form. That is why we present a complete proof.

To deal with the first part we need some algebraic notions.

Definition 3.1. *Let G be a finite group. By a complex group algebra of G we mean the algebra of complex-valued functions on G with the convolution as a multiplication. Its center is denoted by $Z\mathbb{C}[G]$. The set of conjugacy classes of G is denoted by Λ_G .*

The center $Z\mathbb{C}[G]$ of complex group algebra has a natural basis indexed by Λ_G (see, for example [10], appendix A). Namely, if μ is an arbitrary conjugacy class, the corresponding basis element is its characteristic function (here we regard a conjugacy class as a subset of G). For a fixed group G denote the basis element of $Z\mathbb{C}[G]$ corresponding to the conjugacy class μ by f_μ . This basis is referred to as the *standard basis*. The trace function tr_G , endowing $Z\mathbb{C}[G]$ with the Frobenius structure, is defined as follows.

Definition 3.2. *Let G be a finite group. The trace function is a linear functional $\text{tr}_G: Z\mathbb{C}[G] \rightarrow \mathbb{C}$ defined on the elements of the standard basis as follows.*

$$\text{tr}_G(f_\mu) = \begin{cases} 1, & \text{if } \mu = (1), \\ 0, & \text{otherwise,} \end{cases}$$

where (1) denotes the conjugacy class of the unit element in G .

We omit the subscript G in the notation for the trace function when it does not cause an ambiguity.

Definition 3.3. *Let Γ be a graph and G is a finite group. By a coloring of Γ we mean a map $c: E(\Gamma) \rightarrow \Lambda_G$. If Γ is an enhanced graph, for a vertex $v \in V^0(\Gamma)$, edge $e \in E_v$ and coloring c , by $\bar{c}_v(e)$ we mean $c(e)$ if e is oriented in such a way that v is a source for it, and the reciprocal conjugacy class otherwise.*

The procedure of the Section 2 allows to construct a presentation of a fundamental group of a three times punctured sphere with 3 generators and 1 relation: use an enhanced graph D with one 3-valent vertex, three 1-valent ones. Fix such a presentation, and denote by p_e the generator corresponding to the edge $e \in E(D)$. Let G be an arbitrary finite group, and c is a coloring of D . Denote by $N_D(c)$ the set of morphisms $H_{0,3} \rightarrow G$, such that the image of the generator p_e is contained in the conjugacy class $c(e)$ for all $e \in E(D)$.

A tautological corollary of the definition of trace function reads:

Lemma 3.4. *Let G be a finite group, and c is a coloring of an enhanced graph D described above. Then the set of morphisms from $H_{0,3}$ to G , such that the image of the generator p_e corresponding to the edge $e \in E(D)$ is contained in $c(e)$ has the cardinality*

$$\#N_D(c) = \text{tr}\left(\prod_{e \in E(D)} f_{\bar{c}_v(e)}\right),$$

where v denotes the only vertex of D .

Now we are ready to prove the following Lemma.

Lemma 3.5. *Let G be a finite group. Let Γ be an enhanced graph with coloring c . Then the set $N_\Gamma(c)$ of morphisms $H_{g,n} \rightarrow G$, such that the image of a generator p_e corresponding to the edge $e \in E(\Gamma)$ belongs to the conjugacy class $c(e)$, has the cardinality*

$$\#N_\Gamma(c) = |G|^g \left(\prod_{v \in V^0} \text{tr}\left(\prod_{e' \in E_v} f_{\bar{c}_v(e')}\right) \right) \prod_{e \in E^0} \frac{1}{\text{tr}(f_{c(e)} f_{c^{-1}(e)})},$$

where $c^{-1}(e)$ denotes the conjugacy class reciprocal to $c(e)$ for every edge $e \in E(\Gamma)$.

Proof: First of all, note, that for every conjugacy class $\mu \in \Lambda_G$, $\text{tr}(f_\mu f_{\mu^{-1}})$ equals the number of group elements belonging to μ .

Consider the enhanced graph Γ' which is obtained from Γ by cutting all the edges not belonging to the maximal tree and attaching 1-valent vertices to the remaining half-edges. The newly obtained leaves inherit the orientation of edges and the coloring from cut edges. The cyclic orientation in of half-edges adjacent to vertices remains the same. Denote the inherited coloring by c' . The enhanced graph Γ' allows to construct a presentation of $H_{0,g+n}$.

The maximal tree for Γ' coincides with Γ' .

The number $\#N_{\Gamma'}(c')$ is computed inductively. Pick an arbitrary vertex v and make a random choice of images of the generators p_e corresponding to the edges $e \in E_v$. According to Lemma 3.4 it can be made in $\text{tr}(\prod_{e \in E_v} f_{\bar{c}(e)})$ different ways.

Let v' be the second vertex adjacent to e . As the choice of the image for the generator p_e is already made, the choice for the images for the generators corresponding to the remaining edges adjacent to v' can be performed in

$$\frac{\text{tr}(\prod_{e' \in E_{v'}} f_{\bar{c}(e')})}{\text{tr}(f_{c(e)} f_{c^{-1}(e)})}$$

ways.

Any two vertices in Γ' are connected with at most one edge. It implies that

$$\#N_{\Gamma'}(c') = \prod_{v \in V^0(\Gamma)} \text{tr}(\prod_{e' \in E_v} f_{\bar{c}(e')}) \prod_{e \in E^0(\Gamma')} \frac{1}{\text{tr}(f_{c(e)} f_{c^{-1}(e)})}.$$

There is a natural map $b: E(\Gamma') \rightarrow E(\Gamma)$, which sends an edge of Γ' to the corresponding edge of Γ before cut. The map b gives rise to a morphism $\phi: H_{0,g+n} \rightarrow H_{g,n}$. On the level of generators, this morphism is described in the following way:

- If e' is an inner edge of Γ' , or e' is a leaf that is shared by both Γ and Γ' , the generator $p_{e'}$ maps to the generator corresponding to the edge $b(e') \in E(\Gamma)$.
- If e' is a leaf of Γ' came from a cut edge of Γ , and its end is a sink of e' , the generator $p_{e'}$ maps to the generator corresponding to the edge $b(e') \in E(\Gamma)$.
- If e' is a leaf of Γ' came from a cut edge of Γ , and its end is a source of e' , the generator $p_{e'}$ maps to the $g_{b(e')}^{-1} p_{b(e')} g_{b(e')}$.

The check that ϕ is indeed a group morphism is straightforward.

The morphism ϕ induces a natural map $\phi^*: N_\Gamma(c) \rightarrow N_{\Gamma'}(c')$.

For an edge $e \in E(\Gamma)$ that is not contained in the maximal tree denote by e^+ its preimage such that its end is the sink of e^+ . The other preimage is denoted by e^- .

Let $f' \in N_{\Gamma'}(c')$. We can construct out of it a morphism $f \in N_\Gamma(c)$ in the following way. If $e \in \Gamma$ is contained in the maximal tree, the image of $f(p_e)$ coincides with the image $f'(p_{b^{-1}(e)})$. Otherwise, the image $f(p_e)$ coincides with $f'(p_{e^+})$.

The only thing that is left is to fix images of generators g_e corresponding to edges of Γ which are not contained in the maximal tree. We fix them in such a way, that $f'(p_{e^-}) = f(g_e^{-1}) f'(p_{e^+}) f(g_e)$. The check that f is indeed a group morphism is straightforward.

Moreover, it is clear that $\phi^*(f) = f'$. As for every e not belonging to the maximal tree, the choice of the image of g_e can be performed in

$$\frac{|G|}{\text{tr}(f_{c(e)} f_{c^{-1}(e)})}$$

ways, and the choices for different edges can be made completely independently, as each g_e appears in only one relation, we arrive to the assertion of the Lemma.

Q.E.D.

Recall, that in fact we are interested in the number of morphisms $H_{g,n} \rightarrow G$ such that only the conjugacy classes of images of the generators corresponding to the leaves are fixed.

Definition 3.6. Let G be a finite group, Γ be a graph. A boundary condition is a map $M: E(\Gamma) \setminus E^0(\Gamma) \rightarrow \Lambda_G$. The set of all possible colorings $c: E(\Gamma) \rightarrow \Lambda_G$ such that $M = c|_{E(\Gamma) \setminus E^0(\Gamma)}$ is denoted by $C(M)$. Abusing the notation denote $\cup_{C(M)} N_\Gamma(c) = N_\Gamma(M)$.

Theorem 3.7. Let G be a finite group, Γ be an enhanced graph, and M be a boundary condition. Than the number $\#N_\Gamma(M) = \sum_{c \in C(M)} \#N_\Gamma(c)$ is invariant under the following enhanced graphs transformations:

- Other choice of the maximal tree in Γ ;
- Change of orientation of an edge $e \in E^0(\Gamma)$;
- Change of cyclic order of half-edges adjacent to a vertex $v \in V^0(\Gamma)$;
- Local transformation of Γ as it is depicted on the figure 3.

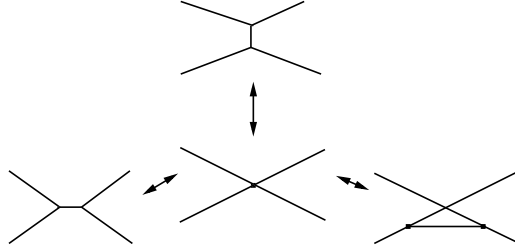


FIGURE 3. The local graph transformations. The figure shows the possible resolutions of a 4-valent vertex.

Proof: A different choice of the maximal tree in Γ respect every single summand in the sum $\sum_{c \in C(M)} \#N_\Gamma(c)$.

Suppose, an edge $e \in E^0(\Gamma)$ has a coloring $c(e)$. A change of the orientation of an inner edge e , and switching its coloring to the reciprocal $c^{-1}(e)$ also preserves every single term in the sum under consideration.

The independence on the choice of the cyclic ordering of half-edges adjacent to a vertex $v \in V^0(\Gamma)$ follows from the commutativity of $Z\mathbb{C}[G]$.

The invariance under the local transformation of Γ follows from the fact that the number of possible morphisms from one group to another (with some fixed conditions) does not depend on the presentation on the group. But we would like to give an alternative argument.

The commutative algebra $Z\mathbb{C}[G]$ can be endowed with a Hermitian product. Namely, consider a semilinear conjugation acting on a basis element f_μ for $\mu \in \Lambda_G$ in the following way:

$$\bar{\cdot}: f_\mu \mapsto f_{\mu^{-1}}.$$

The product is defined by formula $(a, b) = \text{tr}(a\bar{b})$ for $a, b \in Z\mathbb{C}[G]$. The basis vectors of $Z\mathbb{C}[G]$ form an orthogonal basis with respect to this product.

Consider a trace of a product of four basis vectors of $Z\mathbb{C}[G]$ corresponding to conjugacy classes $\mu_1, \mu_2, \mu_3, \mu_4 \in \Lambda_G$. Due to general properties of Hilbert spaces, and

commutativity of $Z\mathbb{C}[G]$, for any permutation $\sigma \in S_4$, this trace can be expanded:

$$\mathrm{tr}(f_{\mu_1} f_{\mu_2} f_{\mu_3} f_{\mu_4}) = \sum_{\nu \in \Lambda_G} \frac{\mathrm{tr}(f_{\mu_{\sigma(1)}} f_{\mu_{\sigma(2)}} f_{\nu}) \mathrm{tr}(f_{\mu_{\sigma(3)}} f_{\mu_{\sigma(4)}} f_{\nu^{-1}})}{\mathrm{tr}(f_{\nu} f_{\nu^{-1}})}.$$

This formula implies the desired invariance.

Q.E.D.

As the space of all connected 1-3-valent graphs is connected with respect to above mentioned transformations (see, for example [9]), the only data that matters for computation of the number $\#N_{\Gamma}(M)$ is a genus of an enhanced graph Γ , the orientation of the leaves of Γ , and boundary condition μ . If we are dealing with a group, where every conjugacy class is a self-reciprocal, like a symmetric group S_n , the orientation of the leaves is also not relevant at all.

Example: Let $\Gamma(2, 1)$ be a connected 1-3-valent graph of genus 2 with 1 leaf, and (-1) be the conjugacy class of the non-unit element of S_2 . Provide Γ with an arbitrary enhancement. We are interested in the computation of the number $\#N_{\Gamma(2,1)}(M)$, where M is a boundary condition which assigns class (-1) to the only leaf of $\Gamma(2, 1)$. As it was stated above, this number is independent on the choice of the enhancement of Γ .

Let c be a coloring of $\Gamma(2, 1)$. The multiplication table of $Z\mathbb{C}[S_2]$ imply that if there exists a vertex $v \in V^0(\Gamma(2, 1))$, such that there are exactly one or three edges adjacent to v colored by (-1) , than this coloring does not contribute to the sum.

By the contrary, if every vertex $v \in V^0(\Gamma(2, 1))$ is adjacent to exactly zero or two edges of coloring (-1) , this coloring gives a contribution 1.

This actually means, that $\#N_{\Gamma(2,1)}(M) = 0$.

From the other hand, for the same group S_2 , and for a graph $\Gamma(2, 2)$ of genus 2 with 2 leaves, and the boundary condition M' which assigns (-1) to every leaf of the graph, the number $\#N_{\Gamma(2,2)}(M')$ is 16, which coincides with the prediction of the Frobenius formula (see, for example [10], appendix A).

4. COMPLEX FEYNMAN CALCULUS

This section can be considered as a short exposition of the theory of the Feynman calculus.

The Feynman calculus is a powerful tool for enumerating graphs. We will apply it to compute generating functions for number of principle G -bundles. Briefly speaking, the machinery of Feynman calculus works as follows. Consider we have a graph with edges colored by a finite number of colors. Feynman calculus provides us with a rule how to assign a number to every vertex and every edge of a such a graph. The weight of a graph is calculated as a product of the above mentioned numbers over all the vertices and edges divided by the order of the automorphism group of the graph. Than we take a sum of weights over all possible graphs. It turns out, that this sum can be interpreted as a result of a computation of an integral. For the details on Feynman calculus in the real case we refer the reader to [6] or [10].

M. Mulase and J. Yu in [11] studied the Feynman calculus over a von Neumann algebra. In fact, our considerations are a generalization of the methods of Mulase and Yu on the case of graphs with 1-valent vertices.

In a sense, our considerations can be treated as a theory of Feynman calculus over a space of diagonal matrices.

Here we will be interested in the complex Feynman calculus. The reason for it is that complex calculus works for any finite group G , while the real calculus works only in the cases when every conjugacy class of G is a self-reciprocal class.

We will need some preliminary considerations to proceed.

Definition 4.1. Let P be a finite-dimensional complex vector space endowed with a Hermitian product $\langle \cdot, \cdot \rangle$. By \mathbb{C}^n we mean an n -dimensional complex vector space with a chosen orthonormal basis $\{f_i\}_{i=1}^n$ with respect to the chosen Hermitian product. The measure dm is defined as $dm = \prod_{i=1}^n d\operatorname{Re} z_i d\operatorname{Im} z_i$, where z_i is the coordinate corresponding to the basis element f_i .

Lemma 4.2. Let A be an endomorphism of \mathbb{C}^n , expressed in a standard basis as a positively definite symmetric real matrix, and p be an arbitrary vector in \mathbb{C}^n . The following equalities are satisfied:

$$(1) \int_{\mathbb{C}^n} e^{-\langle Az, z \rangle} dm = \frac{\pi^n}{\det A};$$

$$(2) \int_{\mathbb{C}^n} e^{-\langle Az, z \rangle + \langle p, z \rangle + \langle z, p \rangle} dm = \frac{\pi^n}{\det A} e^{\langle A^{-1}p, p \rangle}.$$

Proof: The first equality can be deduced from the fact that every sesquilinear positively definite form can be diagonalized by the action of $U(n)$. So it is enough to verify this formula in one dimensional case, where it is nothing else but a product of two Gaussian integrals.

The second equality follows from the first one by the variable change $z \mapsto z + A^{-1}p$.

Q.E.D.

The immediate consequence of the lemma are the equalities:

$$\int_{\mathbb{C}^n} z_i \bar{z}_j e^{-\langle Az, z \rangle} dm = \frac{\pi^n}{\det A} A_{ij}^{-1} \text{ for all } i, j,$$

$$\int_{\mathbb{C}^n} z_i z_j e^{-\langle Az, z \rangle} dm = 0,$$

$$\int_{\mathbb{C}^n} \bar{z}_i \bar{z}_j e^{-\langle Az, z \rangle} dm = 0.$$

They can be obtained by the differentiation of the second equality of the Lemma by p or \bar{p} .

Definition 4.3. Let k be a positive integer, and K_1, K_2 be two finite sets. By a pairing we mean a bijection $\sigma: K_1 \rightarrow K_2$. The set of all pairings is denoted by $\Pi(K_1, K_2)$.

It is clear that $\#\Pi(K_1, K_2) = \delta_{\#K_1, \#K_2} (\#K_1)!$, where $\delta_{\cdot, \cdot}$ is the Kronecker symbol.

The next theorem is a standard fact. Its proof in the real case can be found in [6]. The proof for the complex case can be obtained by the same considerations.

Theorem 4.4 (Wick). Let $L = \{l_1, \dots, l_m\}$, $L' = \{l'_1, \dots, l'_n\}$ be two collections of complex-linear forms on \mathbb{C}^n , and A be an endomorphism of \mathbb{C}^n expressed in the standard basis by a symmetric positively definite real matrix. Then

$$\int_{\mathbb{C}^n} \prod_{l \in L} l(z) \prod_{l' \in L'} \bar{l}'(z) e^{-\langle Az, z \rangle} dm = \frac{\pi^n}{\det A} \sum_{\sigma \in \Pi(L, L')} \prod_{l \in L} \langle A^{-1}l, \sigma(l) \rangle,$$

and the integral converges absolutely.

Let us again stress the fact that the only way to obtain non-zero is to have the same number of linear and semilinear forms under the integral.

5. DOUBLE HURWITZ NUMBERS

Definition 5.1. Let Σ be an oriented surface and $f, g: \Sigma' \rightarrow \Sigma$ be two principle G -bundles with a non-trivial holonomy a finite number of points. Then f and g are considered to be equivalent, if there exists an automorphism $h: \Sigma' \rightarrow \Sigma'$ such that $f = g \circ h$. The automorphism group $\operatorname{Aut} f$ of a principle G -bundle $f: \Sigma' \rightarrow \Sigma$ is

a group of automorphisms $h: \Sigma' \rightarrow \Sigma'$ such that $f = f \circ h$. The number $\frac{1}{\#(\text{Aut } f)}$ is referred to as the weight of the cover.

Consider the following enumerative problem.

Problem: Fix $k + 2$ points q_0, \dots, q_{r+1} on \mathbb{CP}^1 and a finite group G . Choose two conjugacy classes μ and ν of G . Find the weighted number principle G -bundles \mathbb{CP}^1 , with a non-trivial holonomy around q_0, \dots, q_{r+1} , such that the holonomy around q_0 is in μ , the holonomy around q_{r+1} is in ν , and the holonomy around all the other points (q_1, \dots, q_n) belong to a fixed conjugacy class τ .

Denote the corresponding number by $h\tau(\mu, \nu; r)$.

As it was mentioned in the introduction, the weighted number of principle G -bundles with fixed conjugacy class of holonomies, can be computed as a number of the morphisms from the fundamental group of a $r + 2$ -times punctured sphere to the group G divided by the order of G .

Definition 5.2. Let G be a finite group and $\{f_\mu\}_{\mu \in \Lambda_G}$ be the standard basis of $\mathbb{Z}\mathbb{C}[G]$. Endow $\mathbb{Z}\mathbb{C}[G]$ with a Hermitian product $\langle \cdot, \cdot \rangle_G$, such that the basis $\{f_\mu\}_{\mu \in \Lambda_G}$ is orthonormal with respect to it. Let $\tau \in \Lambda_G$ be a fixed conjugacy class. Denote by $A_\tau(\beta)$ a linear endomorphism of $\mathbb{Z}\mathbb{C}[G]$ represented in the standard basis as the matrix with entries

$$(A_\tau)_{\mu, \nu}(\beta) = \text{tr}(f_{\mu^{-1}} f_\nu (1 - \beta f_\tau))$$

for $\mu, \nu \in \Lambda_G$.

Note, that for any τ there exists a neighborhood of the point $\beta = 0$ such that the matrix $A_\tau(\beta)$ is non-degenerate.

Theorem 5.3. Let G be a finite group, and μ, ν and τ be three elements of Λ_G . The generating function $h_{\mu, \nu}^\tau(\beta) = \sum_r h_\tau(\mu, \nu; r) \beta^r$ is given by the following formula:

$$h_{\mu, \nu}^\tau(\beta) = \frac{1}{Z_\tau} \frac{\text{tr}(f_{\mu^{-1}} f_\mu) \text{tr}(f_{\nu^{-1}} f_\nu)}{\#G} \int_{\mathbb{Z}\mathbb{C}[G]} z_\mu \bar{z}_\nu e^{-\langle A_\tau(\beta) z, z \rangle} dm,$$

where Z_τ denotes the value of the integral

$$Z_\tau = \int_{\mathbb{Z}\mathbb{C}[G]} e^{-\langle A_\tau(\beta) z, z \rangle} dm.$$

Proof: Let Γ_r be an enhanced graph of genus 0 with $r + 2$ leaves such that every inner vertex of Γ_r is adjacent to at least one leaf. It is clear, that there is exactly two inner vertices v_1 and v_2 of Γ_r adjacent to two leaves. Denote one of the leaves adjacent to v_1 as e_1 , and one of the leaves adjacent to v_2 as e_2 . Orient the leaves of Γ_r such that for each of them the corresponding end is a sink.

Denote by M_r a boundary condition which assigns the class μ to e_1 , the class ν to e_2 , and class of a permutation τ to every other leaf of Γ_r .

It is clear from our previous considerations that $h_\tau(\mu, \nu, r) = \frac{\#N_{\Gamma_r}(M_r)}{\#G}$.

According to the Theorem 3.7, the number of covers can be computed by the formula:

$$h_\tau(\mu, \nu; r) = \frac{1}{\#G} \sum_{c \in C(M_r)} \left(\prod_{v \in V_0(\Gamma_r)} \text{tr} \left(\prod_{e' \in E_v} f_{\bar{c}_v(e')} \right) \right) \prod_{e \in E_0(\Gamma_r)} \frac{1}{\text{tr}(f_{c(e)} f_{c^{-1}(e)})}.$$

Let k be a non-negative integer. Denote by X_r the set of maps

$$x: \{1, \dots, r\} \rightarrow (\Lambda_{S_d})^2.$$

The notation $x_i(k)$ for $i = 1, 2$ denotes the corresponding component of the map. By an automorphism of a map $x \in X_r$ we mean an automorphism h of the set $\{1, \dots, r\}$ such that $x = x \circ h$. The group of automorphisms of $x \in X_r$ is denoted by $\text{Aut } x$.

Let D be a linear automorphism of $Z\mathbb{C}[G]$ represented in a standard basis by a matrix $D_{\mu,\nu} = \text{tr}(f_{\mu^{-1}} f_{\nu})$.

Let us discuss the denominator Z_{τ} in the statement of the Theorem. Using the absolute convergence of the integral in a neighborhood of the point $\beta = 0$, we expand the expression in the following manner:

$$Z_{\tau} = \sum_{r=0}^{\infty} \beta^r \sum_{x \in X_r} \frac{1}{\# \text{Aut } x} \int_{Z\mathbb{C}[G]} \prod_{k=1}^r \text{tr}(f_{x_1(k)} f_{x_2(k)} f_{\tau}) z_{x_1(k)} \bar{z}_{x_2(k)} e^{-\langle Dz, z \rangle} dm.$$

Apply theorem 4.4 for the computation of this expression. For $x \in X_r$ let Π^x denote a subset of $\Pi_r = \Pi(\{1, \dots, r\}, \{1, \dots, r\})$, such that for any $\sigma \in \Pi^x$ $x_1(k) = x_2^{-1}(\sigma(k))$. The elements of Π^x are called *admissible pairings*. For any non-negative integer r , we obtain

$$Z_{\tau} = \frac{\pi^{\#\Lambda_G}}{\det D} \sum_{r=0}^{\infty} \beta^r \sum_{x \in X_r} \sum_{\sigma \in \Pi^x} \frac{1}{\# \text{Aut } x} \prod_{k=1}^r \frac{\text{tr}(f_{x_1(k)} f_{x_2(k)} f_{\tau})}{\text{tr}(f_{x_1(k)} f_{x_2^{-1}(\sigma(k))})}.$$

This expression can be interpreted in the following way. Let a *flower* be an elementary piece of a graph consisting of two vertices connected by an edge and two oriented half-edges attached to one of the vertices. For one of the half-edges the adjacent vertex is a sink (we call this half-edge *left*), for another one the adjacent vertex is a source (this half-edge is *right*). The edge connecting the vertices of a flower is oriented, in such a way that its end is a sink. Each half-edge of a flower is colored by an element of Λ_G , the only edge of a flower is colored by the fixed class τ of G .

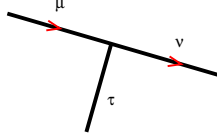


FIGURE 4. A flower. The left half-edge is colored by $\mu \in \Lambda_G$, the right half-edge is colored by $\nu \in \Lambda_G$.

Every pairing $\sigma \in \Pi_r$ corresponds to an element of the symmetric group S_r . Namely, set the result of application of this element to $k \in \{1, \dots, r\}$ to be equal $\sigma(k)$. By the abuse of notation, denote a pairing and the corresponding permutation by the same letter.

Pick an arbitrary $x \in X_r$. For every $k \in \{1, \dots, r\}$ prepare a flower with the left half-edge colored by $x_1(k)$, and the right half-edge colored by $x_2(k)$. Each admissible pairing provides a recipe to assembly a colored 1-3-valent graph. Namely, the right half-edge of the k th flower gets glued to the left half-edge of the $\sigma(k)$ th flower. The resulting graph can be disconnected, but each of its connected components has genus 1. The number of connected components of the graph equals the number of cycles in the permutation σ . For an example see the figure 5.

We claim, that Theorem 3.7 implies that Z_{τ} is a generating function for the weighted number of principle G -bundles over a collection of disjoint tori, such that every non-trivial holonomy around a point is in the class τ multiplied by $\pi^{\#\Lambda_G} / \det D$. Note, that from the other hand, $Z_{\tau} = \pi^{\#\Lambda_G} / \det A_{\tau}(\beta)$.

Now consider the integral

$$Z = \int_{Z\mathbb{C}[G]} z_{\mu} \bar{z}_{\nu} e^{-\langle A_{\tau}(\beta) z, z \rangle} dm.$$

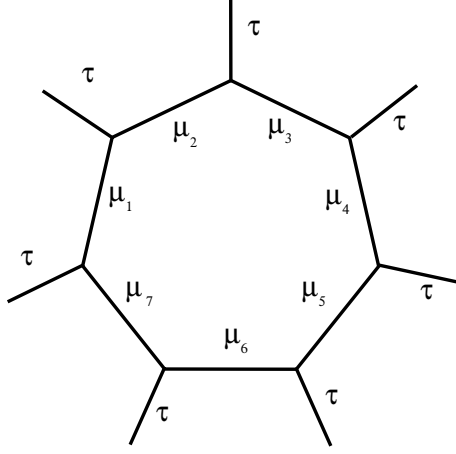


FIGURE 5. An example of assembling a graph. The directions of the edges are not shown.

As it was mentioned during the study of Z_τ , we can use the expansion:

$$\sum_{r=0}^{\infty} \beta^r \sum_{x \in X_r} \frac{1}{\#Aut \ x} \int_{Z\mathbb{C}[G]} z_\mu \bar{z}_\nu \prod_{k=1}^r \text{tr}(f_{x_1(k)} f_{x_2(k)} f_\tau) z_{x_1(k)} \bar{z}_{x_2(k)} e^{-\langle Dz, z \rangle} dm.$$

The only difference from Z_τ are two distinguished forms z_μ and \bar{z}_ν . Applying the Theorem 4.4 again, we see, that the computation of this integral differs from the computation of Z_τ only by the existence of two additional elementary blocks. One is represented by a vertex with an adjacent half-edge oriented in such a way that the vertex is a source, colored by μ . The other is represented by a vertex with an adjacent half-edge oriented in such a way that the vertex is a sink, colored by ν . Our rules for the assembling graphs out of flowers and this two additional details are the same. For each graph we pick only one copy of each of these details.

Note, that both distinguished details always contribute to the same connected component of a graph of genus 0. This connected component is exactly the graph we discussed in the beginning of the proof.

The division by Z_τ allows to get rid of the contribution of all the components of genus 1. The origin of the factor $\text{tr}(f_\mu^{-1} f_\mu) \text{tr}(f_\nu^{-1} f_\nu) / \#G$ is obvious.

Q.E.D.

The direct application of the theorem 4.4 lead to the following corollary:

Corollary 5.4.

$$h_{\mu, \nu}^\tau(\beta) = \frac{\text{tr}(f_\mu^{-1} f_\mu) \text{tr}(f_\nu^{-1} f_\nu)}{\#G} (A_\tau^{-1}(\beta))_{\mu, \nu},$$

where $A_\tau^{-1}(\beta)$ is the matrix inverse to $A_\tau(\beta)$.

As the matrix $A_\tau(\beta)$ is linear in β , we obtain:

Corollary 5.5. *For any finite group G and any $\mu, \nu, \tau \in \Lambda_G$ the generating function $h_{\mu, \nu}^\tau(\beta)$ is a rational function in β .*

Another simple theorem follows from our representation of these numbers in the terms of graphs.

Theorem 5.6. *The generating functions for numbers $h_\tau(\mu, \nu; r)$ satisfy the following equation*

$$h_{\mu, \nu}^\tau(\beta) + \beta \frac{\partial}{\partial \beta} h_{\mu, \nu}^\tau(\beta) = (\#G) \sum_{\lambda \in \Lambda_{S_d}} \frac{h_{\mu, \lambda}^\tau(\beta) h_{\lambda, \nu}^\tau(\beta)}{\text{tr}(f_{\lambda^{-1}} f_\lambda)}.$$

This equation corresponds to presentation of a genus 0 chain-like graph as a union of two graphs of the same type along their leaves.

6. DOUBLE HURWITZ NUMBERS

Let d be a positive integer, and $\tau_d \in \Lambda_{S_d}$ be the class of a transposition. In this section we present matrices $A_{\tau_d}(\beta)$ and $(A_{\tau_d}(\beta))^{-1}$ for $d = 2, 3, 4$.

For $d = 2$ the basis elements are ordered in the following manner. The first one corresponds to partition (11), the second one corresponds to (2).

$$A_{\tau_2}(\beta) = \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \quad (A_{\tau_2}(\beta))^{-1} = \begin{pmatrix} \frac{1}{1-\beta^2} & \frac{\beta}{1-\beta^2} \\ \frac{\beta}{1-\beta^2} & \frac{1}{1-\beta^2} \end{pmatrix}.$$

For $d = 3$ we use the ordering (111), (12), (3).

$$A_{\tau_3}(\beta) = \begin{pmatrix} 1 & -3\beta & 0 \\ -3\beta & 3 & -6\beta \\ 0 & -6\beta & 2 \end{pmatrix} \quad (A_{\tau_3}(\beta))^{-1} = \begin{pmatrix} \frac{6\beta^2-1}{9\beta^2-1} & -\frac{\beta}{9\beta^2-1} & -\frac{3\beta^2}{9\beta^2-1} \\ -\frac{\beta}{9\beta^2-1} & -\frac{1}{27\beta^2-3} & -\frac{\beta}{9\beta^2-1} \\ -\frac{3\beta^2}{9\beta^2-1} & -\frac{\beta}{9\beta^2-1} & \frac{3\beta^2-1}{9\beta^2-1} \end{pmatrix}$$

For $d = 4$ the ordering of basis elements is (1111), (112), (22), (13), (4).

$$A_{\tau_4}(\beta) = \begin{pmatrix} 1 & -6\beta & 0 & 0 & 0 \\ -6\beta & 6 & -6\beta & -24\beta & 0 \\ 0 & -6\beta & 3 & 0 & -12\beta \\ 0 & -24\beta & 0 & 8 & -24\beta \\ 0 & 0 & -12\beta & -24\beta & 6 \end{pmatrix}$$

And $(A_{\tau_4}(\beta))^{-1}$ is

$$\begin{pmatrix} \frac{24\beta^4-34\beta^2+1}{144\beta^4-40\beta^2+1} & -\frac{20\beta^3-\beta}{144\beta^4-40\beta^2+1} & \frac{24\beta^4+2\beta^2}{144\beta^4-40\beta^2+1} & -\frac{3\beta^2}{36\beta^2-1} & \frac{16\beta^3}{144\beta^4-40\beta^2+1} \\ -\frac{20\beta^3-\beta}{144\beta^4-40\beta^2+1} & -\frac{20\beta^2-1}{864\beta^4-240\beta^2+6} & \frac{12\beta^3+\beta}{432\beta^4-120\beta^2+3} & -\frac{\beta}{72\beta^2-2} & \frac{8\beta^2}{432\beta^4-120\beta^2+3} \\ \frac{24\beta^4+2\beta^2}{144\beta^4-40\beta^2+1} & \frac{12\beta^3+\beta}{432\beta^4-120\beta^2+3} & \frac{72\beta^4-30\beta^2+1}{432\beta^4-120\beta^2+3} & -\frac{3\beta^2}{36\beta^2-1} & \frac{24\beta^3-2\beta}{432\beta^4-120\beta^2+3} \\ -\frac{3\beta^2}{36\beta^2-1} & -\frac{\beta}{72\beta^2-2} & -\frac{3\beta^2}{36\beta^2-1} & \frac{12\beta^2-1}{288\beta^2-8} & -\frac{\beta}{72\beta^2-2} \\ \frac{16\beta^3}{144\beta^4-40\beta^2+1} & \frac{8\beta^2}{432\beta^4-120\beta^2+3} & -\frac{24\beta^3-2\beta}{432\beta^4-120\beta^2+3} & -\frac{\beta}{72\beta^2-2} & -\frac{20\beta^2-1}{864\beta^4-240\beta^2+6} \end{pmatrix}$$

For example, the generating function

$$h_{(4),(4)}^{\tau_4}(\beta) = \frac{6 \cdot 6}{4!} \frac{1 - 20\beta^2}{864\beta^4 - 240\beta^2 + 6} = \sum_{k=0}^{\infty} \left(\frac{36^k}{8} + \frac{4^{k-1}}{2} \right) \beta^{2k}.$$

Let us try to compare this generating function with the known result. Double Hurwitz numbers with $\mu = (d)$ are known as one-part double Hurwitz numbers. Consider the following function

$$\varsigma(z) = e^{z/2} - e^{-z/2} = 2 \sinh(z/2).$$

A formula for one-part double Hurwitz numbers was derived in. Considering the particular case $\mu = \nu = (d)$, we use for the comparison a formula derived in [8] and reproved in [7]. In our notation, this formula reads:

$$\sum_{r=0}^{\infty} \frac{z^r}{r!} h^{\tau_4}((d), (d); r) = \frac{1}{d^2} \frac{\varsigma(d^2 z)}{\varsigma(dz)}.$$

Substituting $d = 4$, and applying the inverse Borel transform,

$$\frac{1}{4^2} \int_0^\infty e^{-t} \frac{\varsigma(16zt)}{\varsigma(4zt)} dt$$

we obtain a perfect agreement with our result.

REFERENCES

- [1] Atiyah, M. F.: *The geometry and physics of knots*, Lezioni Lincee [Lincei Lectures], Cambridge University Press, (1990).
- [2] Bertrand, B., Brugallé, E., Mikhalkin, G.: *Open Tropical Hurwitz numbers*, Rend. Semin. Mat. Univ. Padova **125** (2011), 157–171.
- [3] Cavalieri, R., Johnson, P., Markwig, H.: *Tropical Hurwitz numbers*, J. Algebr. Comb. **32** (2010), 241–265.
- [4] Dijkgraaf, R.: *Mirror symmetry and elliptic curves*, in C. Faber, R. Dijkgraaf and G. van der Geer, editors, *The Moduli spaces of curves*, Birkhausen, 1995.
- [5] Ekedahl, T., Lando, S., Shapiro, M., Vainshtein, A.: *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. **146** (2001), 297–327.
- [6] Etingof, P.: *Mathematical ideas and notions of quantum field theory*, lecture notes (2002). Available online on <http://www-math.mit.edu/etingof>.
- [7] Johnson, P.: *Double Hurwitz numbers via the infinite wedge*, preprint, arXiv:1008.3266.
- [8] Goulden, I., Jackson, D. M., Vakil, R.: *Towards the geometry of double Hurwitz numbers*, Adv. Math. **198** (2005), 43–92.
- [9] Hatcher, A.: *On triangulation of surfaces*, Topology and its Applications **40** (1991), 189–194.
- [10] Lando, S., Zvonkin, A.: *Graphs on surfaces and their applications. With an appendix by Don B. Zagier*. Encyclopaedia of Mathematical Sciences, 141. Low-dimensional topology, II. Springer-Verlag, Berlin, 2004.
- [11] Mulase, M., Yu, J. T.: *Non-commutative matrix integrals and representation varieties of surface groups in a finite group*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 6, 2161–2196.
- [12] Okounkov, A.: *Toda equations for Hurwitz numbers*, Math. Res. Lett. **7** (2000), no. 4, 447–453.

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